Methods in Calculus Cheat Sheet

Edexcel Core Pure 2

The aim of this chapter is to extend upon the calculus you learnt in Pure Year 2. We will look at improper integrals, the differentiation and integration of inverse trigonometric functions and the use of partial fractions to integrate fractional expressions with quadratic factors in the denominator. Proficiency with the differentiation and integration techniques you covered in Pure Year 2 will be key in this chapter.

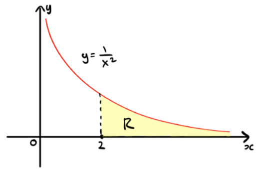
Defining improper integrals

You need to know what is meant by an improper integral, and when they can be evaluated.

- The integral $\int_a^b f(x)\,dx$ is improper if:
  - one or both limits are infinite.
  - $f(x)$ is undefined at some point in the interval $a \le x \le b$.

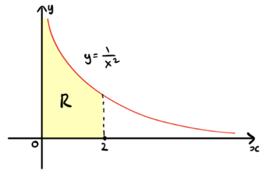
$a \le x \le b$ could also be written as $[a, b]$

If an improper integral exists, we say it is convergent. Otherwise we say it is divergent.



The region R is represented by the improper integral $\int_2^\infty \frac{1}{x^2}\,dx$

This is a improper integral because the upper limit of the integral is positive infinity.



The region R is represented by the improper integral $\int_0^2 \frac{1}{\sqrt{x}}\,dx$

This is an improper integral because the range of integration $[0,2]$ contains the point $x = 0$, where the curve is undefined since $x = 0$ is an asymptote.

Integrals where one of the limits is infinite

When evaluating integrals with infinity in one of the limits, we use a process involving limits:

1. Replace the infinity with another variable (e.g. t)
2. Evaluate the integral as usual.
3. Consider the limit of your result as $t \to \pm\infty$ (this will depend on whether the range of integration included positive or negative infinity).

The limit of the result will give you the final answer. It is important to note that this limit will not always exist. In such cases, the integral is said to be divergent and cannot be evaluated. If the limit does exist however, the integral is convergent and we can evaluate it.

Example 1: Evaluate the integral $\int_2^\infty \frac{1}{x^2}\,dx$, or show that it is not convergent.

Replacing the infinity with t	$\int_2^t \frac{1}{x^2}\,dx$
Carrying out the integration:	$\left[-\frac{1}{x}\right]_2^t = \left[-\frac{1}{t} + \frac{1}{2}\right]$
Finding the limit of our result as $t \to \infty$ Note that as $t \to \infty, -\frac{1}{t} \to 0$.	$\lim_{t\to\infty}\left[-\frac{1}{t} + \frac{1}{2}\right] = \frac{1}{2}$

Integrals where both limits are infinite

If both limits are infinite, then you need to split the integral up into the sum of two improper integrals as follows:

- $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{k} f(x)\,dx + \int_{k}^{\infty} f(x)\,dx$

You can choose any value of k when splitting up such an integral. Remember that if both integrals converge then the original will converge. If one diverges however, then the original will diverge.

Integrals that are undefined at some point

When dealing with integrals where the integrand is undefined at some point in the given interval, we use a very similar limiting process. If the undefined point is one of your limits of integration:

1. Identify which point, k, in the interval is undefined and replace this with another variable, t.
2. Evaluate the integral as usual.
3. Consider the limit of your result as $t \to k$. This will be your final answer.

In the rare occasion that the undefined point is not one of your limits, you will need to split the integral up before using the above process:

- If $f(x)$ is undefined at $x = k$ over the interval $[a, b]$, then $\int_a^b f(x)\,dx = \int_a^k f(x)\,dx + \int_k^b f(x)\,dx$

Example 2: Evaluate the integral $\int_0^2 \frac{1}{\sqrt{x}}\,dx$, or show that it is not convergent.

Notice that the function is undefined at $x = 0$, so we replace 0 with t and integrate.	$\int_t^2 \frac{1}{\sqrt{x}}\,dx = [2\sqrt{x}]_t^2 = [2\sqrt{2} - 2\sqrt{t}]$
Finding the limit of our result as $t \to 0$. Note that as $t \to 0,\ 2\sqrt{t} \to 0$.	$\lim_{t\to 0}[2\sqrt{2} - 2\sqrt{t}] = 2\sqrt{2} + 0 = 2\sqrt{2}$

The mean value of a function

You already know that to calculate the mean of a set of values, you must sum up the values and divide by the number of values. We will now extend this idea to look at how we find the mean value of a function over a given interval.

- The mean value, $\bar{f}$, of a function $f(x)$ over the interval $a \le x \le b$ is given by $\frac{1}{b-a}\int_a^b f(x)\,dx$.

You can use the following rules to deduce the mean value of some transformed functions:

If the function $f(x)$ has mean value $\bar{f}$ over the interval $a \le x \le b$, then:

- $f(x) + k$ has mean value $\bar{f} + k$ over the interval $a \le x \le b$,
- $kf(x)$ has mean value $k\bar{f}$ over the interval $a \le x \le b$,
- $-f(x)$ has mean value -$\bar{f}$ over the interval $a \le x \le b$,

where k is a real constant.

Example 3: Find the exact mean value of $f(x) = \frac{sinxcosx}{cos2x+4}$ over the interval $\left[0, \frac{\pi}{2}\right]$.

Using the mean value formula:	$\bar{f} = \frac{1}{\frac{\pi}{2} - 0}\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\cos 2x + 4}\,dx$
Rewriting the numerator using the identity $\sin 2x = 2\sin x \cos x$	$\frac{1}{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\frac{1}{2}\sin 2x}{\cos 2x + 4}\,dx$
Simplifying:	$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\cos 2x + 4}\,dx$
Using the reverse chain rule to integrate, then evaluating our result using the given limits:	$= \frac{1}{\pi}\left[-\frac{1}{2}\ln\lvert\cos 2x + 4\rvert\right]_0^{\frac{\pi}{2}} = \frac{1}{\pi}\left[-\frac{1}{2}\ln\lvert 3\rvert + \frac{1}{2}\ln\lvert 5\rvert\right]$ $= \frac{1}{2\pi}\ln\left\lvert\frac{5}{3}\right\rvert$

Differentiating inverse trigonometric functions

You need to be able to differentiate expressions involving the inverse trigonometric functions. The following results are useful, and you could be asked to prove them:

- $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$

To prove these results, you should use implicit differentiation. Follow the same technique used in part (i) of example 4.

It is important to note that you cannot use these results to **directly** differentiate transformations of the inverse functions. For example, you cannot conclude that $\frac{d}{dx}(\arcsin 2x) = \frac{1}{\sqrt{1-(2x)^2}}$. If you wish to differentiate such expressions, you must use either implicit differentiation or the chain rule. You should use whichever method you prefer. Here is an example showing both methods in action:

Example 4: Given that $y = \arctan(x^2)$, find $\frac{dy}{dx}$ using: (i) implicit differentiation (ii) the chain rule.

(i) Taking tan of both sides:	$y = \arctan(x^2) \Rightarrow \tan y = x^2$
Differentiating both sides with respect to x (implicitly):	$\sec^2 y\frac{dy}{dx} = 2x$
Making $\frac{dy}{dx}$ the subject:	$\frac{dy}{dx} = \frac{2x}{\sec^2 y}$
We want $\frac{dy}{dx}$ in terms of x only, so we need to find a way to write sec^2y in terms of x. Using the identity $1 + tan^2y \equiv sec^2y$:	$\frac{dy}{dx} = \frac{2x}{1+\tan^2 y}$
Since $tany = x^2,\ \ tan^2y = x^4$.	$\frac{dy}{dx} = \frac{2x}{1+x^4}$
(ii) We start by replacing the input with another variable t:	Let $t = x^2$, then $y = \arctan t$
Differentiating with respect to t. We use the result $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$	$\frac{dy}{dt} = \frac{1}{1+t^2}$
We use the chain rule to figure out what else we need to find $\frac{dy}{dx}$.	$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$
Since we let $t = x^2$ at the beginning, we use this to find $\frac{dt}{dx}$.	$t = x^2 \Rightarrow \frac{dt}{dx} = 2x \ \therefore \frac{dy}{dx} = \frac{2x}{1+t^2}$
Using $t = x^2$ to express $\frac{dy}{dx}$ solely in terms of x:	$\frac{dy}{dx} = \frac{2x}{1+(x^2)^2} = \frac{2x}{1+x^4}$

Integrating with inverse trigonometric functions

You also need to know and be able to prove the following results:

- $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$ (I)
- $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + c$ (II)
- $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert + c$ (III)

To prove this result, you need to first rewrite the integral as $\int \frac{1}{(a+x)(a-x)}\,dx$, then use partial fractions to integrate the expression in the same way you learnt to do in Pure Year 2 (Chapter 11).

These results are given to you in the formula booklet, and you are allowed to use them without proof unless of course you are explicitly asked to prove them. The following example contains the proof for results (I) and (II).

Example 5: Show that: (i) $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$. (ii) $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + c$.

(i)	
Using the substitution $x = a\tan u$:	$x = a\tan u$ $\frac{dx}{du} = a\sec^2 u \Rightarrow dx = a\sec^2 u\,du$
Our integral becomes:	$\int \frac{1}{a^2 + a^2\tan^2 u}a\sec^2 u\,du$ $= \int \frac{1}{a^2(1+\tan^2 u)}a\sec^2 u\,du$
Simplifying using $1 + tan^2u \equiv sec^2u$	$= \int \frac{1}{a^2\sec^2 u}a\sec^2 u\,du = \int \frac{1}{a}du$
Carrying out the integration:	$= \frac{u}{a} + c$
Finding u in terms of x so we can write our result in terms of x:	Since $x = a\tan u$, $\frac{x}{a} = \tan u$ $\therefore u = \arctan\left(\frac{x}{a}\right)$
So our result is:	$\frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$

(ii)	
Using the substitution $x = asinu$:	$x = a\sin u$ $\frac{dx}{du} = a\cos u \Rightarrow dx = a\cos u\,du$
Our integral becomes:	$\int \frac{1}{\sqrt{a^2 - a^2\sin^2 u}}a\cos u\,du$ $= \int \frac{1}{\sqrt{a^2(1-\sin^2 u)}}a\cos u\,du$
Simplifying:	$= \int \frac{1}{a\cos u}a\cos u\,du = \int 1\,du$
Carrying out the integration:	$= u + c$
Finding u in terms of x so we can write our result in terms of x:	Since $x = a\sin u$, $\frac{x}{a} = \sin u$ $\therefore u = \arcsin\left(\frac{x}{a}\right)$
So our result is:	$\arcsin\left(\frac{x}{a}\right) + c$

In general, the procedure to integrate functions which are similar to (I) and (II) is to manipulate the expression to match one of the forms (I) or (II), then use their results to evaluate the integral. However, you may need to use a substitution if you can't manipulate into one of these forms. The following tips are helpful for such cases:

- For integrals involving $a^2 + x^2$, try the substitution $x = a\tan u$.
- For integrals involving $\sqrt{a^2 - x^2}$, try the substitution $x = a\sin u$.

Integrating using partial fractions with non-linear factors

Previously in Pure Year 2, you learnt how to use partial fractions to integrate fractions where one or more linear factors were present in the denominator. You also need to know how to use partial fractions to integrate expressions where there are one or more quadratic factors of the form $ax^2 + b$ in the denominator. The partial fractions method does not change but we have to make a small modification to the first step of the process:

- When using partial fractions with a quadratic factor in the denominator, you must ensure the numerator of the quadratic fraction is in a linear form (i.e. $ax + b$).

Here are three examples outlining how we begin the partial fractions procedure with quadratic factors:

One quadratic factor	Multiple quadratic factors	Repeated quadratic factor
$\frac{2}{(x^2+1)(x+3)} \equiv \frac{Ax+B}{(x^2+1)} + \frac{C}{(x+3)}$	$\frac{x^3+9x^2+x-1}{x^4-1} \equiv \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2-1)}$	$\frac{x^4+1}{x(x^2+1)^2} \equiv \frac{A}{x} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2}$

Example 6: Find $I = \int \frac{2}{(x^2+1)(x+1)}\,dx$.

Setting up the partial fractions, making the denominators the same and equating the numerators:	$\frac{2}{(x^2+1)(x+1)} \equiv \frac{Ax+B}{(x^2+1)} + \frac{C}{(x+1)}$ $\Rightarrow 2 \equiv (Ax+B)(x+1) + C(x^2+1)$
To find A, B, C we can compare coefficients: We could also use the substitution method to find A, B, C.	$2 = B + C$ (comparing constants) $A + B = 0$ (comparing x coefficients) $A + C = 0$ (comparing x^2 coefficients) $\Rightarrow A = -1, B = 1, C = 1$
Our integral becomes:	$\int \frac{-x+1}{(x^2+1)} + \frac{1}{(x+1)}\,dx$
To integrate the first expression, we split the fraction into two, and use the reverse chain rule to integrate the first fraction and the result (I) for the second.	$\int \frac{-x+1}{(x^2+1)}\,dx = \int \frac{-x}{(x^2+1)} + \frac{1}{(x^2+1)}\,dx$ $= -\frac{1}{2}\ln(x^2+1) + \arctan x + c$
To integrate the second expression, we use a result from Pure Year 2.	$\int \frac{1}{(x+1)}\,dx = \ln\lvert x+1\rvert + c$
Putting our answers together:	$\therefore I = -\frac{1}{2}\ln(x^2+1) + \ln\lvert x+1\rvert + \arctan x + c$
Simplifying use log rules:	$= \ln\left\lvert\frac{x+1}{\sqrt{x^2+1}}\right\rvert + \arctan x + c$

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